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Hardi - Volterra integral operatori normasi uchun aniq qiymat

Точное значение нормы оператора Харди-Вольтерра

The exact value for the norm of the Hardy – Volterra integral operator

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Annotatsiya. Ushbu ishda vaznli Lebeg fazosida berilgan yadrosi o‘zgarmas bo‘lgan Hardi-Volterra operatori normasi uchun aniq qiymat keltirilgan.
Kalit so‘zlar: Hardi-Volterra operatori, Hardi tengsizligi, vaznli Lebeg fazosi, operator normasi, vazn.
Abstract. In this paper the exact value is given for the norm of the Hardy – Volterra operator of the constant kernel in the weighted Lebesgue space.
Keywords: Hardy - Volterra operator, Hardy inequality, kernel, weighted Lebesgue space, norm, weight.
Аннотация. В данной работе представлено точное значение нормы оператора Харди-Вольтерра константного ядра, заданного в весовом пространстве Лебега.
Ключевая слова: Операторы Харди-Волтерра, неравенство Харди, ядро, весовые пространства Лебега, норма оператора, весь.

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Kirish

Ta'rif 1.1 *Vaznli Lebeg fazosi deb*

$$\|f\|_{s,w} := \left(\int_a^b |f|^s w(x)dx \right)^{\frac{1}{s}} < \infty, \; 1 \le s \le \infty,$$

shartni qanoatlantiradigan (a,b) *dagi barcha o'lchovli funksiyalardan iborat fazoga aytiladi va u quyidagicha belgilanadi*

$$L_{s,w}(a,b) = L_{s,w},$$

bunda $w - (a,b)$ *dagi vazn funksiyasi, ya'ni berilgan intervalda o'lchovli, nomanfiy funksiya.*

Buyuk Britaniyalik olim G.H.Hardi 1920 yilda quyidagi teoremani isbotladi.

Teorema 1.1 (Hardi teoremasi, [2].) *Faraz qilaylik* $f(x) \ge 0$, $f -$ *chekli,* $(0,X)$ *oraliqda va* $f^2 - (0,\infty)$ *da integrallanuvchi bo'lsa, u holda quyidagi tengsizlik o'rinli bo'ladi*

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$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^2 dx \le 4\int_0^\infty f^2(x)dx, \qquad (1.1)$$

bunda 4 soni tengsizlikning eng yaxshi konstantasidir, ya'ni tengsizlik o'rinli bo'ladigan sonlarning eng kichigidir.

Agar

$$Hf(x) = \int_0^x f(t)dt \qquad (1.2)$$

ko'rinishda Hardi-Volterra integral operatorini qarasak u holda (1.1) ni quyidagi ko'rinishda ham yozish mumkin:

$$\left\|Hf\right\|_{2,x^{-2}} \le 2\left\|f\right\|_2,$$

bunda

$$\left\|Hf\right\|_{2,x^{-2}} = \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^2 \frac{1}{x^2}dx\right)^{\frac{1}{2}},$$

$$\left\|f\right\|_2 = \left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{2}}.$$

Yuqoridagi tengsizlik va Hardi teoremasi shuni ko'rsatadiki H operatorning normasi uchun quyidagi tenglik o'rinli bo'lar ekan

$$\left\|H\right\|_{L_2 \to L_{2,x^{-2}}} = 2.$$

Umuman olganda vazn funksiyani ixtiyoriy o'zgartirish bilan operator normasini topish, bu juda murakkab masala. Hattoki vazn funksiya ixtiyoriy ratsional funksiya bo'lgan holda ham bu masala ochiq hisoblanadi. Lekin bunday masalalar differensial va integral operatorlarning asosiy xossalarini o'rganishda juda muhim hisoblanandi.

Ushbu maqolada vazn funksiyasiga biroz o'zgartirish kiritilishi operatorning normasiga tasir ko'rsatmasligi o'rganilgan. Maqola kirish qismi, asosiy natijalar va foydalanilgan adabiyotlar ro'yxatidan iborat.

Asosiy natijalar

Ushbu paragrafda biz (1.2) Hardi – Volterra operatorini ushbu holda qaraymiz

$$H: L_2(0,1) \to L_2\left((0,1), \frac{1}{\left(x+x^2\right)^2}\right).$$

Ishning asosiy natijasi quyidagi teoremadir:

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Teorema 2.1. *Hardi-Volterra integral operatori normasi uchun quyidagi baho o'rinlidir*

$$\left\| H(f) \right\|_{L_2\left((0,1),\frac{1}{(x+x^2)^2}\right)} \leq 2\left\| f \right\|_{L_2(0,1)} \tag{1.3}$$

bu yerda

$$\left\| H \right\|_{L_2\left((0,1),\frac{1}{(x+x^2)^2}\right)\to L_2(0,1)} = 2.$$

Isbot. Yuqoridagi (2.1) tengsizlikni unga ekvivalent ko'rinishda quyidagicha yozish mumkin:

$$\int_0^1 \left(\frac{1}{x+x^2} \int_0^x f(t)dt \right)^2 dx \leq 4\int_0^1 f^2(x)dx. \tag{1.4}$$

(2.2) tengsizlikning chap qismiga avval Gyolder tengsizligini so'ngra Fubini teoremasini qo'llaymiz

$$\left\| H f \right\|^2_{L_2\left((0,1),\frac{1}{(x+x^2)^2}\right)} = \int_0^1 \left(\frac{1}{x+x^2} \int_0^x f(t)dt \right)^2 dx$$

$$= \int_0^1 \left(\frac{1}{x+x^2} \int_0^x f(t)t^{\beta}t^{-\beta}dt \right)^2 dx$$

$$\leq \int_0^1 \left(\int_0^x f^2(t)\, t^{\frac{1}{2}}dt \right)\left(\int_0^x t^{-\frac{1}{2}}dt \right) \frac{1}{(x+x^2)^2} dx$$

$$= \int_0^1 f^2(t)\, t^{\frac{1}{2}} \left[\int_t^1 \left(\int_0^x t^{-\frac{1}{2}}dt \right) \frac{1}{(x+x^2)^2} dx \right] dt$$

$$=2\int_0^1 f^2(t)\, t^{\frac{1}{2}} \left[\int_t^1 \frac{x^{\frac{1}{2}}}{(x+x^2)^2} dx \right] dt$$

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$$= 2\int_0^1 f^2 \;\; t \;\; t^{\frac{1}{2}} \left[\int_t^1 \frac{x^{-\frac{3}{2}}}{1+x^{2}} dx \right] dt = J$$

$\frac{1}{(1+x)^2}$ funksiya kamayuvchi ekanligidan o'zining $(t,1)$ intervalda eng katta qiymatiga $x = t$ da erishishini e'tiborga olsak quyidagi bahoga ega bo'lamiz:

$$J \le 2\int_0^1 f^2 \;\; t \;\; \frac{t^{\frac{1}{2}}}{1+t^{2}} \left[\int_t^1 x^{-\frac{3}{2}} dx \right] dt$$

$$= 2\int_0^1 f^2 \;\; t \;\; \frac{t^{\frac{1}{2}}}{1+t^{2}} \frac{1-t^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)} dt = I$$

Bunda $\frac{1}{1+t^{2}}$ funksiya kamayuvchi ekanligidan

$$I \le 4\int_0^1 f^2 \;\; t \;\; \frac{1}{1+t^{2}} dt$$

$$\le 4\int_0^1 f^2 \;\; t \;\; dt = 4\left\|f\right\|^2_{L_2(0,1)}$$

demak

$$\left\|H \quad f\right\|^2_{L_2\left((0,1),\frac{1}{(x+x^2)^2}\right)} \le 4\left\|f\right\|^2_{L_2 \quad 0,1}$$

kelib chiqadi. Bundan berilgan integral operator chegaralangan va uning normasi uchun ushbu tengsizlik o'rinli bo'lishi kelib chiqadi

$$\left\|H\right\|_{L_2\left((0,1),\frac{1}{(x+x^2)^2}\right) \to L_2(0,1)} \le 2. \qquad (1.5)$$

Endi operator normasi 2 ekanligini ko'rsatish uchun (2.3) ga teskari bahoni, ya'ni

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$$\|H\|_{L_2\left((0,1),\frac{1}{(x+x^2)^2}\right)\to L_2(0,1)} \geq 2$$

bahoni ko‘rsatamiz. Buning uchun (2.2) tengsizlikda quyidagi test funksiyani olamiz

$$f_\tau(t) = \begin{cases} t^\alpha, & 0 < t < \tau \\ 0, & \tau < t \end{cases},$$

bunda $\alpha > -\frac{1}{2}$, u holda (2.2) tengsizlikning chap tomondan bahosini ham ko‘rsatish mumkin.

$$\int_0^\tau \left(\frac{1}{x+x^2}\int_0^x t^\alpha dt\right)^2 dx$$

$$+\int_\tau^1 \left(\frac{1}{x+x^2}\int_0^x t^\alpha dt\right)^2 dx \leq C\int_0^\tau x^{2\alpha}dx$$

$$\frac{1}{(1+\alpha)^2}\int_0^\tau \left(\frac{x^{1+\alpha}}{x+x^2}\right)^2 dx$$

$$+\frac{\tau^{2(1+\alpha)}}{(1+\alpha)^2}\int_\tau^1 \frac{1}{(x+x^2)^2}dx \leq C\frac{\tau^{1+2\alpha}}{1+2\alpha}$$

tengsizlikning har ikkala tomonini $\frac{\tau^{1+2\alpha}}{1+2\alpha}$ ga bo‘lamiz

$$\frac{1+2\alpha}{(1+\alpha)^2\tau^{1+2\alpha}}\int_0^\tau \left(\frac{x^{1+\alpha}}{x+x^2}\right)^2 dx + \frac{\tau(1+2\alpha)}{(1+\alpha)^2}\int_\tau^1 \frac{1}{(x+x^2)^2}dx \leq C. \tag{1.6}$$

(2.4) ning chap qismidagi ifodaning $\tau = 0$ nuqta atrofidagi xususiyatini o‘rganamiz. Buning uchun bu ifodaning $\tau \to 0$ dagi limitini Lapital qoidasi bilan hisoblaymiz:

$$\lim_{\tau\to 0}\frac{\int_0^\tau \left(\frac{x^{1+\alpha}}{x+x^2}\right)^2 dx}{\tau^{1+2\alpha}} = \lim_{\tau\to 0}\frac{\left(\frac{\tau^\alpha}{1+\tau}\right)^2}{(1+2\alpha)\tau^{2\alpha}} = \frac{1}{1+2\alpha} \tag{1.7}$$

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$$\lim_{\tau\to 0}\tau\int_{\tau}^{1}\frac{1}{(x+x^2)^2}dx=\lim_{\tau\to 0}\frac{\int_{\tau}^{1}\frac{1}{(x+x^2)^2}dx}{\frac{1}{\tau}}=\lim_{\tau\to 0}\frac{\frac{1}{(\tau+\tau^2)^2}}{\frac{1}{\tau^2}}=1.$$

(1.8)

(2.5) va (2.6) ni (2.4) ga qo‘yib quyidagini ifodani hosil qilamiz

$$\frac{1}{(1+\alpha)^2}+\frac{(1+2\alpha)}{(1+\alpha)^2}\le C$$

yoki

$$\frac{2}{1+\alpha}\le C$$

$\alpha>-\frac{1}{2}$ dagi supremumini topamiz

$$\sup_{\alpha>-\frac{1}{2}}\frac{2}{1+\alpha}=4$$

demak

$$\left\|H\right\|^2_{L_2\left((0,1),\frac{1}{x+x^2}{}^2\right)\to L_2(0,1)}\ge 4$$

ya’ni

$$\left\|H\right\|_{L_2\left((0,1),\frac{1}{x+x^2}{}^2\right)\to L_2(0,1)}\ge 2. \qquad (1.9)$$

(2.3) va (2.7) dan

$$\left\|H\right\|_{L_2\left((0,1),\frac{1}{x+x^2}{}^2\right)\to L_2(0,1)}=2$$

ni olamiz. Bu esa (2.1) tengsizlikning to‘liq isbotlanganini bildiradi. Teorema isbot bo’ldi.

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Xulosa. Umuman olganda yuqoridagi natijalarni $s = 2$ dan farqli boshqa qiymatlarda ham qarash mumkin edi. Bunda operator normasi $\frac{s}{s-1}$ ga teng bo'ladi. Qaralgan vazn funksiyaning asosiy xususiyatlaridan biri shuki, vazn funksiya berilgan intervalning nol nuqtasida maxsuslikka ega bo'lib, uning cheksizlikka intilish tartibi $\frac{1}{x^2}$ operator normasi o'zgarmasligi uchun muhim o'rin tutadi.

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