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IKKI O'LCHAMLI PANJARADAGI SISTEMALARGA MOS DISKRET SHREDINGER OPERATORLARI XOS QIYMATLARI VA ULARNING ASIMPTOTIKALARI

АСИМПТОТИКА СОБСТВЕННЫХ ЗНАЧЕНИЙ ДИСКРЕТНЫХ ОПЕРАТОРОВ ШРЕДИНГЕРА, АССОЦИИРОВАННЫХ С ДВУХЧАСТОТНЫМИ СИСТЕМАМИ НА РЕШЕТКЕ

ASYMPTOTICS OF EIGENVALUES OF RANK-ONE PERTURBATIONS OF THE DISCRETE SCHRÖDINGER OPERATOR ON TWO-DIMENSIONAL LATTICE

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Abstract. We consider the family $H_v(\mu)$ of discrete Schrödinger-type operator in two-dimensional lattice $\mathbb{T}^2$. We establish the existence or non-existence and also the finiteness of eigenvalues of $H_v(\mu)$ lying above the essential spectrum. Moreover, we study the properties of eigenvalues as a function of μ, in particular, we find the asymptotics of eigenvalues as sufficiently small and positive μ.

Keywords. *operator, spectrum, eigenvalue, eigenvalue asymptotic.*

Annotatsiya. Biz ikki o'lchamli panjarada $H_v(\mu)-$ bir zarrachali diskret Schrödinger operatorini o'rganamiz. $H_v(\mu)$ operatorning muhim spektrdan yuqorida xos qiymatlari mavjudligi yoki mavjud emasligini va sonini aniqlaymiz. Bundan tashqari, xos qiymat funksiyasining xossalarini va yetarlicha kichik, musbat μ larda xos qiymatning asimptotikasini o'rganamiz.

Kalit so'zlar. *operator, spektr, xos qiymat, xos qiymat asimptotikasi.*

Аннотация. Мы исследуем одночастичный дискретный оператор Шредингера$H_v(\mu)$на одномерной решетке $\mathbb{T}^2$. Установим существование или несуществование, а также конечность собственных значений оператора $H_v(\mu)$ лежащих ниже существенного спектра. Кроме того, мы изучаем свойства собственных значений в зависимости от μ, в частности, находим асимптотику собственных значений при достаточно малом и положительном μ.

Ключевые слова. *оператор, спектр, собственное значение, асимптотика.*

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1.INTRODUCTION

In [3] Klaus studied the discrete spectrum of the Schrödinger operator $-\frac{d^2}{dx^2}+\lambda V$ for $\lambda>0$ and V obeying

$$\int_{\mathbb{R}}(1+|x|)|V(x)|dx<\infty,$$

extending the results of Simon in [2] in $d=1$. Klaus showed that if $\int V(x)dx>0$, then for small and positive λ there is no bound state, and if $\int V(x)dx\leq 0$, then there exists a bound state $E(\lambda)$ and it satisfies

$$(-E(\lambda))^{\frac{1}{2}}=-\frac{\lambda}{2}\int V(x)dx-\frac{\lambda^2}{4}\int V(x)|x-y|V(y)dxdy+o(\lambda^2)$$

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as $\lambda \searrow 0$.

The family $H_v(\mu)$ given by below can be seen as a discrete counterpart of a continuous nonlocal Schrödinger-type operators appearing as a diffusion generator in certain stochastic models (see e.g. [2–6]). At the same time, they can also be seen as an effective one-particle discrete Schrödinger operator $H_v(\mu; K)$ parametrized by quasi-momenta $K \in T\ d$, associated to the Hamiltonian of a system of two arbitrary and/or identical particles in d-dimensional lattice Z^d(see e.g. [4–11]).

Such lattice models have become popular in recent years because they represent a minimal, natural Hamiltonian describing the systems of ultracold atoms in optical lattices – systems with highly controllable parameters such as lattice geometry and dimensionality, particle masses, tunneling, temperature etc. (see e.g., [7,8,9,10] and references therein). In contrast to usual condensed matter systems, where stable composite objects are usually formed by attractive forces and repulsive forces separate particles in free space, the controllability of collision properties of ultracold atoms allowed to observe experimentally a stable repulsive bound pair of ultracold atoms in the optical lattice Z^3, see e.g., [1,4,7].

2. DISCRETE SHRODINGER OPERATOR

In the momentum space representation, the operator acts in $L^2(\mathbb{T}^2)$ by

$$H_v(\mu) := H_0 + \mu V_v, \quad \mu > 0.$$

The free Hamiltonian H_0 is the multiplication operator in $L^2(\mathbb{T}^2)$ by the function

$$\varepsilon := 2\pi\mathcal{F}\hat{\varepsilon}$$

so-calledthe *dispersionrelation* ofthe particle andthe potential$V_{a,b}$ acts on$L^2(\mathbb{T}^2)$ as a convolution operator

$$V_v f(p) = v(p)\int_{\mathbb{T}^2} v(q) f(q) \mathrm{d}q$$

definedin$L^2(\mathbb{T}^2)$,where $v(\cdot)$is a givennonzero real-analyticfunction on$\mathbb{T}^2$. By the clasic Weyl theorem we obtain

$$\sigma_{\mathrm{ess}}\big(H_v(\mu)\big) = \sigma(H_0) = [\varepsilon_{\min}, \varepsilon_{\max}], \qquad \mu > 0,$$

where

$$\varepsilon_{\min} := \min \varepsilon, \varepsilon_{\max} := \max \varepsilon.$$

Inwhatfollowswe always assume:

Hypothesis 1 *The dispersionrelation$\varepsilon$is a real-valued evenfunction, symmetricwithrespectto coordinate permutations andhaving a non-degenerate unique maximum at$\vec{\pi} = (\pi, \pi) \in \mathbb{T}^2$.Moreover, $\varepsilon$is analyticnear$\vec{\pi}$.*

Remark 1 *In view of Hypothesis 1*

$$\nabla\varepsilon(\vec{\pi}) = \left(\frac{\partial \varepsilon}{\partial q_1}(\vec{\pi}), \frac{\partial \varepsilon}{\partial q_2}(\vec{\pi})\right) = (0,0)$$

and the Hessian

$$\nabla^2\varepsilon(\vec{\pi}) = \begin{pmatrix} \dfrac{\partial^2 \varepsilon}{\partial q_1^2}(\vec{\pi}) & \dfrac{\partial^2 \varepsilon}{\partial q_1 \partial q_2}(\vec{\pi}) \\ \dfrac{\partial^2 \varepsilon}{\partial q_1 \partial q_2}(\vec{\pi}) & \dfrac{\partial^2 \varepsilon}{\partial q_2^2}(\vec{\pi}) \end{pmatrix}$$

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is strictly negative definite. By the symmetricity of ε

$$\frac{\partial^2\varepsilon}{\partial q_1^2}(\vec{\pi}) = \frac{\partial^2\varepsilon}{\partial q_2^2}(\vec{\pi}) < 0 \quad and \quad \left|\frac{\partial^2}{\partial q_1^2}(\vec{\pi})\right| > \left|\frac{\partial^2}{\partial q_1 \partial q_2}(\vec{\pi})\right|.$$

Moreover, bythe Morse lemma there exist a neighborhood$U(\vec{\pi}) \subset \mathbb{T}^2$ *of*$\vec{\pi} \in \mathbb{T}^2$ *and an analyticdiffeomorphism*$\psi: B_\gamma(0) \to U(\vec{\pi})$,*where* $B_\gamma(0) \subset \mathbb{R}^2$*is a discin*$\mathbb{R}^2$ *ofradius*$\gamma \in (0,1)$*centered atthe origin, suchthat*$\psi(0) = \vec{\pi}$ *and*

$$\varepsilon(\psi(y)) = \varepsilon_{max} - y^2, \quad y \in B_\gamma(0).$$

Moreover, the Jacobian $J(\psi(y))$ *of* ψ *is strictly positive in* $B_\gamma(0)$. *We write*

$$J_0 := \frac{1}{4\pi} J(\psi(0)) > 0.$$

3. MAIN RESULTS AND PROOFS

In this section we study theeigenvalues of operators by the min-max principle, $H_v(\mu)$ can haveat most oneeigenvalueoutside theessential spectrum. Also note that $z_0 > \varepsilon_{\max}$ is eigenvalueof $H_v(\mu)$ if and only if z_0 is a zeroof the corresponding Fredholm determinant

$$\Delta_v(\mu; z) := 1 - \mu \int_{\mathbb{T}^2} \frac{v(q)^2 \mathrm{d}q}{z - \varepsilon(q)}.$$

Such an equivalence will be frequently used subsequently. First we study the existence of eigenvalues of $H_v(\mu)$.

Theorem 1. *Let* v *be nonzero real-analytic function on* $\mathbb{T}^2$ *and let*

$$\mu_v^0 := \left[\int_{\mathbb{T}^2} \frac{v(q)^2 dq}{\varepsilon_{max} - \varepsilon(q)}\right]^{-1}.$$

Then$\mu_v^0 = 0$*if*$v(\vec{\pi}) \neq 0$ *and*$\mu_v^0 \in (0, +\infty)$*if*$v(\vec{\pi}) = 0$.*Moreover, for any*$\mu > \mu_v^0$*the operator*$H_v(\mu)$*has a unique eigenvalue* $z_v(\mu) > \varepsilon_{max}$ *andthe corresponding eigenfunctionis*

$$f_\mu(p) = \frac{v(p)}{z_v(\mu) - \varepsilon(p)}.$$

Moreover, the function $z_v(\cdot)$ *is real-analytic, strictly increasing and strictly convex in* $(\mu_v^0, +\infty)$ *with the asymptotics*

$$z_v(\mu) \searrow \varepsilon_{max} \quad as \quad \mu \searrow \mu_v^0, \tag{1}$$

and

$$z_v(\mu) = \mu \int_{\mathbb{T}^2} v(q)^2 \, dq \; + \left[\int_{\mathbb{T}^2} v(q)^2 \, dq\right]^{-1} \int_{\mathbb{T}^2} v(q)^2 \varepsilon(q) dq \; + O\left(\frac{1}{\mu}\right)$$

as $\mu \to +\infty$. *Finally, if* $v \in L^{2,\omega}(\mathbb{T}^2)$ *for some* $\omega \in \{os, oa, ea, es\}$, *then* $L^{2,\omega}(\mathbb{T}^2)$ *is invariant subspace of* $H_v(\mu)$ *and* $f_\mu \in L^{2,\omega}(\mathbb{T}^2)$.

We drop the proof since the all assertions but the last one can be done along the lines of [1, Theorem 2.1] using Proposition 1. The last assertion is obvious.

Next we study the asymptotics of $z_v(\mu)$ as $\mu \searrow \mu_v^0$.

Theorem 2.

(a) Assume that $v(\vec{\pi}) \neq 0$. *Then for sufficiently small and positive* $\mu > 0$

$$z_v(\mu) = \varepsilon_{max} + c_v e^{-\frac{1}{4\pi^2 J_0 v(\vec{\pi})^2 \mu}} + \sum_{\substack{n,m \geq 0, \\ n+m \geq 1}} c_{nm} \mu^n \left(\frac{1}{\mu} e^{-\frac{1}{4\pi^2 J_0 v(\vec{\pi})^2 \mu}}\right)^{m+1}, \tag{2}$$

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where $J_0 > 0$ *is given by Remark 3.1,* $\{c_{nm}\}$ *are real coefficients,*

$$c_v := e^{\frac{\omega_\varepsilon}{4\pi^2 J_0 v(\vec{\pi})^2}}$$

and $\omega_\varepsilon \in \mathbb{R}$ *is a constant depending only on* ε;

(b) Assume that $v(\vec{\pi}) = 0$ *and* $\nabla v(\vec{\pi}) \neq 0$. *Then for sufficiently small and positive* $\mu - \mu_v^0$

$$z_v(\mu) = \varepsilon_{max} + \frac{c_v\,(\mu-\mu_v^0)}{-ln(\mu-\mu_v^0)} + \sum_{\substack{n,m,k\geq 0,\\ n+m+k\geq 1}} c_{nmk}(\mu-\mu_v^0)^n \left(\frac{\mu-\mu_v^0}{-ln(\mu-\mu_v^0)}\right)^{m+1}\left(\frac{lnln(\mu-\mu_v^0)^{-1}}{-ln(\mu-\mu_v^0)}\right)^k, (3)$$

where $\{c_{nmk}\}$ *are realcoefficients and*

$$c_v = \frac{1}{2\pi^2 J_0 (\mu_v^0)^2}\left[\left(\frac{\partial v}{\partial q_1}(\vec{\pi})\right)^2\left(\frac{\partial \psi}{\partial y_1}(\vec{0})\right)^2 + \left(\frac{\partial v}{\partial q_2}(\vec{\pi})\right)^2\left(\frac{\partial \psi}{\partial y_2}(\vec{0})\right)^2\right]^{-1};$$

(c) Assume that$v(\vec{\pi}) = 0$ *and*$\nabla v(\vec{\pi}) = 0$.*Thenforsufficientlysmall andpositive* $\mu - \mu_v^0$

$$z_v(\mu) = \varepsilon_{max} + c_v(\mu-\mu_v^0) + \sum_{\substack{n,m\geq 0,\\ n+m\geq 1}} c_{nm}(\mu-\mu_v^0)^{n+1}(-(\mu-\mu_v^0)ln(\mu-\mu_v^0))^m, \qquad (3^*)$$

where $\{c_{nm}\}$ *are real coefficients and*

$$c_v = \frac{1}{(\mu_v^0)^2}\left[\int_{\mathbb{T}^2}\frac{v(q)^2\,dq}{(\varepsilon_{max}-\varepsilon(q))^2}\right]^{-1} > 0.$$

Proof. By the Theorem 1 $\mu_v^0 = 0$ if $v(\vec{\pi}) = 0$and $\mu_v^0 > 0$ if $v(\vec{\pi}) \neq 0$. Recall that $z := z_v(\mu) > \varepsilon_{\max}$ is an eigenvalueof H_μ^v if and only if its Fredholm determinant Δ^v satisfies

$$\Delta_v(\mu; z) = 1 - \mu\int_{\mathbb{T}^2}\frac{v(q)^2 \mathrm{d}q}{z-\varepsilon(q)} = 0. \qquad (4)$$

By (1) $z \searrow \varepsilon_{\max}$ as $\mu \searrow \mu_v^0$. Applied with $v := v^2$ for sufficiently small $\mu - \mu_v^0 > 0$ (so that $z - \varepsilon_{\max} > 0$ is also small) the equation (4) is represented as follows:

(a) if $v(\vec{\pi}) \neq 0$,

$$\frac{1}{\mu} = \omega_0^{(2)} - 4\pi^2 J_0 v(\vec{\pi})^2 \ln(z-\varepsilon_{\max}) + \sum_{n\geq 1}\omega_n^{(1)}(z-\varepsilon_{\max})^n \ln(z-\varepsilon_{\max})) + \sum_{n\geq 1}\omega_n^{(2)}(z-\varepsilon_{\max})^n; \qquad (5)$$

(b) if $v(\vec{\pi}) = 0$ and $\nabla v(\vec{\pi}) \neq 0$,

$$\frac{1}{\mu} - \frac{1}{\mu_v^0} = \sum_{n\geq 1}\omega_n^{(1)}(z-\varepsilon_{\max})^n \ln(z-\varepsilon_{\max}) + \sum_{n\geq 1}\omega_n^{(2)}(z-\varepsilon_{\max})^n, \qquad (6)$$

where

$$\omega_1^{(1)} = 2\pi^2 J_0\left[\left(\frac{\partial v}{\partial q_1}(\vec{\pi})\right)^2\left(\frac{\partial \psi}{\partial y_1}(\vec{0})\right)^2 + \left(\frac{\partial v}{\partial q_2}(\vec{\pi})\right)^2\left(\frac{\partial \psi}{\partial y_2}(\vec{0})\right)^2\right] > 0;$$

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(c) if $v(\vec{\pi}) = 0$ and $\nabla v(\vec{\pi}) = 0$,

$$\frac{1}{\mu} - \frac{1}{\mu_v^0} = \sum_{n\geq 1} \omega_n^{(2)}(z-\varepsilon_{\max})^n + \sum_{n\geq 2} \omega_n^{(1)}(z-\varepsilon_{\max})^n \ln(z-\varepsilon_{\max}), \quad (7)$$

where

$$\omega_1^{(2)} = -\int_{\mathbb{T}^2} \frac{v(q)^2 \mathrm{d}q}{(\varepsilon_{\max} - \varepsilon(q))^2} < 0.$$

Note thatthe coefficients$\{\omega_n^{(1)}\}, \omega_n^{(2)}\}$in (5)-(7) are realnumbersdepending only onε.Setting

$$\lambda = \mu - \mu_v^0 \text{and} \quad \alpha = z - \varepsilon_{\max},$$

we represent (5) as

$$\frac{1}{\mu} = \omega_0^{(2)} - 4\pi^2 J_0 v(\vec{\pi})^2 \ln\alpha + \sum_{n\geq 1} \omega_n^{(1)} \alpha^n \ln\alpha + \sum_{n\geq 1} \omega_n^{(2)} \alpha^n, \quad (8)$$

(6) as

$$-\frac{\lambda}{\mu_v^0 \lambda + (\mu_v^0)^2} = \sum_{n\geq 1} \omega_n^{(1)} \alpha^n \ln\alpha + \sum_{n\geq 1} \omega_n^{(2)} \alpha^n \quad (9)$$

and (7) as

$$-\frac{\lambda}{\mu_v^0 \lambda + (\mu_v^0)^2} = \sum_{n\geq 1} \omega_n^{(2)} \alpha^n + \sum_{n\geq 2} \omega_n^{(1)} \alpha^n \ln\alpha. \quad (10)$$

(a) To find the implicit function $\alpha = \alpha(\mu)$ solving (8) we set

$$\alpha := e^{-\frac{1}{4\pi^2 J_0 v(\vec{\pi})^2 \mu}} (u + d_0), \tau := \frac{1}{\mu} e^{-\frac{1}{4\pi^2 J_0 v(\vec{\pi})^2 \mu}}, \quad (11)$$

where

$$d_0 := e^{\frac{\omega_0^{(2)}}{4\pi^2 J_0 v(\vec{\pi})^2}} > 0$$

and τ is small if $\mu > 0$ is small. Inserting this change of variables in (8) we get

$$\begin{aligned} F(u,\mu,\tau) &:= \omega_0^{(2)} - 4\pi^2 J_0 v(\vec{\pi})^2 \ln(u + d_0) \\ &+ \sum_{n\geq 1} \omega_n^{(1)} \tau^n (u+d_0)^n \left(\frac{\mu^{n-1}}{4\pi^2 J_0 v(\vec{\pi})^2} + \mu^n \ln(u+d_0) \right) \\ &+ \sum_{n\geq 1} \omega_n^{(2)} \mu^n \tau^n (u+d_0)^n = 0. \end{aligned}$$

The function $F(u,\mu,\tau)$ is real-analytic for small $|u|, |\mu|, |\tau|$ and by the definition of d_0,

$$F(0,0,0) = 0 \text{and} \frac{\partial F}{\partial u}(0,0,0) = -\frac{4\pi^2 J_0 v(\vec{\pi})^2}{d_0} < 0.$$

Thenbythe implicitfunctiontheoreminthe analyticalcase forsufficientlysmall$|\mu|$ and$|\tau|$there exists a unique $u = u(\mu,\tau)$solving$F(u,\mu,\tau) \equiv 0$ andisgivenbythe absolutelyconvergentseries

$$u = \sum_{n,m\geq 0} c_{nm} \mu^n \tau^m, \quad (12)$$

where $\{c_{nm}\}$ are real coefficients. Since $u(0) = 0, c_{00} = 0$. Inserting the representation (12) of u in the expression of α in (11) we get (1).

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Before solving (9) and (10) in α first we observe that the function $\lambda := \lambda(\cdot)$ is continuous and satisfies

$$\lambda(0) := \lim_{\alpha \searrow 0} \lambda(\alpha) = 0.$$

By continuity, there exists $\alpha_1 > 0$ such that $\lambda(\alpha) \in (0, \mu_v/2)$ for all $\alpha \in (0, \alpha_1)$. Thus, for such α we can write

$$\frac{\lambda}{(\mu_v^0)^2(1+\lambda/\mu_v^0)} = \frac{\lambda}{(\mu_v^0)^2}\sum_{n\geq 0}(-1)^n\frac{\lambda^n}{(\mu_v^0)^n}, \quad (13)$$

thus,

$$\frac{\lambda}{(\mu_v^0)^2} + \sum_{n\geq 1}\frac{(-1)^n\lambda^{n+1}}{(\mu_v^0)^{n+2}} = \omega_1^{(1)}\alpha\ln\alpha + \omega_1^{(2)}\alpha + \sum_{n\geq 2}\left(\omega_n^{(1)}\alpha^n\ln\alpha + \omega_n^{(2)}\alpha^n\right), \alpha \in (0, \alpha_1). \quad (14)$$

(b) Since $\omega_1^{(1)} > 0$ in (9), setting

$$\alpha := \tau\left(\frac{1}{\omega_1^{(1)}(\mu_v^0)^2} + u\right), \quad \tau := -\frac{\lambda}{\ln\lambda}, \quad \sigma := -\frac{\ln\ln\lambda^{-1}}{\ln\lambda},$$

in (9) and using (13), as in (a) we get (2).

(c) Since$\omega_1^{(2)} < 0$ in (10), setting

$$\alpha := \lambda\left(-\frac{1}{\omega_1^{(2)}(\mu_v^0)^2} + u\right), \quad \theta := -\lambda\ln\lambda,$$

in (10) and using (13), as in (a) we get (3).■

Finally we study the threshold resonances and threshold eigenfunctions.

Theorem 3. *Let* $v(\vec{\pi}) = 0$ *and*

$$f_v(p) = \frac{v(p)}{\varepsilon_{max} - \varepsilon(p)}.$$

- *Let* $\nabla v(\vec{\pi}) \neq 0$. *Then* $f_v \in L^1(\mathbb{T}^2)\backslash L^2(\mathbb{T}^2)$.
- *Let* $\nabla v(\vec{\pi}) = 0$. *Then* $f_v \in L^2(\mathbb{T}^2)$.

Proof. Since both $v(p)^2$ and $\varepsilon(q) - \varepsilon_{\max}$ behave like $(p - \vec{\pi})^2$ near $\vec{\pi}$, repeating a similar argument to Proposition 1 with $v = |v|$ we get

$$\int_{\mathbb{T}^2}|f_v|\mathrm{d}p = \lim_{z\to\varepsilon_{\max}}\int_{\mathbb{T}^2}\frac{|v(p)|\mathrm{d}p}{z-\varepsilon(p)} < +\infty,$$

hence, $f_v \in L^1(\mathbb{T}^2)$.Nowif$\nabla v(\vec{\pi}) \neq 0$,then$v(p)^2$behaveslike $(p-\vec{\pi})^2$ and$(\varepsilon(q) - \varepsilon_{\max})^2$behaveslike $(p-\vec{\pi})^4$near$\vec{\pi}$,hence,

$$\int_{\mathbb{T}^2}|f_v|^2\mathrm{d}p = \lim_{z\to\varepsilon_{\max}}\int_{\mathbb{T}^2}\frac{|v(p)|\mathrm{d}p}{z-\varepsilon(p)} = +\infty,$$

hence, $f_v \notin L^2(\mathbb{T}^2)$.Finally, if$\nabla v(\vec{\pi}) = 0$,thenboth$v(p)^2$ and$(\varepsilon(q) - \varepsilon_{\max})^2$behave like $(p-\vec{\pi})^4$near$\vec{\pi}$ andhence,

$$\int_{\mathbb{T}^2}|f_v|^2\mathrm{d}p = \lim_{z\to\varepsilon_{\max}}\int_{\mathbb{T}^2}\frac{|v(p)|^2\mathrm{d}p}{(z-\varepsilon(p))^2} < +\infty$$

so that$f_v \in L^2(\mathbb{T}^2)$. ■

4. APPENDIX. ASYMPTOTIC EXPANSION OF SOME INTEGRALS

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Proposition 1. *Let ε be continuous a real-valued even function, symmetric with respect to coordinate permutations and having a non-degenerate unique maximum at $\vec{\pi} \in \mathbb{T}^d, d = 1,2$. Moreover, assume that ε is analytic near $\vec{\pi}$. Then, thereexists $\delta := \delta(v, \varepsilon) \in (0,1)$ such that for any $z \in (\varepsilon_{max}, \varepsilon_{max} + \delta)$ the function $B(z)$ is represented as follows:*

• *if $d = 1$,*

$$B(z) = \frac{\pi J(\psi(0))}{\sqrt{z - \varepsilon_{max}}} v(\pi) + \sum_{j \geq 0} \omega_j (z - \varepsilon_{max})^{\frac{j}{2}}, \tag{15}$$

where $\{\omega_j\}$ are real numbers and power series converges absolutely;

• *if $d = 2$,*

$$B(z) = -\pi J(\psi(0)) v(\vec{\pi}) ln(z - \varepsilon_{max}) + \ln(z - \varepsilon_{max}) \sum_{j \geq 1} \omega_j^{(1)} (z - \varepsilon_{max})^j + \sum_{j \geq 0} \omega_j^{(2)} (z - \varepsilon_{max})^j \tag{16}$$

where $\{\omega_j^{(1)}\}$ and$\{\omega_j^{(2)}\}$ are realnumbers andbothpowerseriesconverges absolutely.

Proof. Accordingto the hypothesis,

$$\nabla \varepsilon(\vec{\pi}) = 0 \text{and} \nabla^2 \varepsilon(\vec{\pi}) < 0.$$

Moreover, bythe Morse lemma there exist a neighbourhood$U(\vec{\pi}) \subset \mathbb{T}^d$ of$\vec{\pi} \in \mathbb{T}^d$ and an analyticdiffeomorphism$\psi: B_\gamma(0) \to U(\vec{\pi})$,where $B_\gamma(0) \subset \mathbb{R}^d$is a discin$\mathbb{R}^d$ ofradius$\gamma \in (0,1)$centered atthe origin, suchthat$\psi(0) = \vec{\pi}$ and

$$\varepsilon(\psi(y)) = \varepsilon_{\max} - y^2$$

for all $y \in B_\gamma(0)$. Furthermore, the Jacobian $J(\psi(y))$ of map ψ is strictly positive in $B_\gamma(0)$. Then given $z > \varepsilon_{\max}$,

$$B(z) = \int_{U(\vec{\pi})} \frac{v(q) \mathrm{d}q}{z - \varepsilon(q)} + \int_{\mathbb{T}^d \setminus U(\vec{\pi})} \frac{v(q) \mathrm{d}q}{z - \varepsilon(q)} =: I_1(z) + I_2(z). \tag{17}$$

Since $\vec{\pi}$ is the unique maximum of $\varepsilon, I_2(\cdot)$ is an analytic function of z in a (complex) neighbourhood of $\varepsilon_{\max}$ so that

$$I_2(z) = \sum_{k \geq 0} (-1)^k \int_{\mathbb{T}^d \setminus U(\vec{\pi})} \frac{v(q) \mathrm{d}q}{(\varepsilon_{\max} - \varepsilon(q))^{k+1}} (z - \varepsilon_{\max})^k. \tag{18}$$

Its radius of convergence r_1 depends only on ε (through $U(\vec{\pi})$) and v.

In $I_1(z)$ we first make the change of variables $q \mapsto \psi(y)$ and then use the coarea formula to get

$$I_1(z) = \int_{B_\gamma(0)} \frac{v(\psi(y)) J(\psi(y)) \mathrm{d}y}{y^2 - \varepsilon_{\max} + z}$$

$$= \int_0^\gamma \frac{\mathrm{d}r}{r^2 + z - \varepsilon_{\max}} \int_{\partial B_r(0)} v(\psi(y)) J(\psi(y)) \mathrm{d}\mathcal{H}^{d-1}(y), \tag{19}$$

where $\partial B_r(0) \subset \mathbb{R}^d$isthe sphere ofradius$r > 0$centred atthe origin and$\mathcal{H}^{d-1}$isthe $d - 1$dimensionalHausdorffmeasure – the length ofthe sphere $\partial B_r(0)$. By the Pizzetti formula[1] (see e.g. [4])

[1] The expression of the surface integral mean related to the Laplacian in $\mathbb{R}^n$ as a series in *"radius-square"* whose coefficients are, up to certain dimensional factors, the iterated of the Laplacian itself.

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$$\frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}r^{d-1}}\int_{\partial B_r(0)} v(\psi(y))J(\psi(y))\mathrm{d}\mathcal{H}^{d-1}(y)=\sum_{n\ge 0}C_n r^{2n}, \qquad (20)$$

where

$$C_n:=c_{n,d}\Delta_y^n\left(v(\psi(y))J(\psi(y))\right)|_{y=0}, n\ge 0, \qquad (21)$$

with Δ_y^n being the Laplace operator in $y\in\mathbb{R}^d$ and

$$c_{n,d}:=\frac{\Gamma(d/2)}{4^n n!\,\Gamma(n+d/2)}$$

andΔ_y^nisthe Laplace operatorin$y\in\mathbb{R}^2$.Since v,ψ and$J$ are analyticfunctions, taking$\gamma>0$smallerifnecessary, we assume thatthe seriesin (17) convergesuniformlyin$r\in[0,\gamma]$.Note thatby (18) for$n=0$

$$C_0:=v(\psi(0))J(\psi(0))=v(\vec{\pi})J(\psi(0)). \qquad (22)$$

Inserting (17) in (18) we obtain

$$I_1(z)=\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\sum_{n\ge 0}C_n\int_0^\gamma\frac{r^{2n+d-1}\mathrm{d}r}{r^2+z-\varepsilon_{\max}}, \qquad (23)$$

where the series is uniformly convergent in $r\in[0,\gamma]$. By [5, Lemma B.1], for any integer $n\ge 0$ and $s\in(0,\gamma/2)$,

$$\int_0^\gamma\frac{r^m\mathrm{d}r}{r^2+s}=\begin{cases}\frac{\pi(-s)^{m/2}}{2\sqrt{s}}+\hat{I}_m(s) & \text{if } m \text{ is even},\\ -\frac{1}{2}(-s)^{(m-1)/2}\ln(s)+\tilde{I}_m(s) & \text{if } m \text{ is odd},\end{cases}$$

where $\hat{I}_m$ and$\tilde{I}_m$ are real-analyticfunctionsin$W_1:=\{z\in\mathbb{C}:|z|<1\}$.Moreover, itiswell-knownthat

$$\frac{2\pi^{d/2}}{\Gamma(d/2)}=\begin{cases}2, & \text{if } d=1,\\ 2\pi, & \text{if } d=2.\end{cases}$$

Therefore,

$$I_1(z)=\frac{\pi}{\sqrt{z-\varepsilon_{\max}}}\sum_{n\ge 0}C_n(\varepsilon_{\max}-z)^n+\sum_{n\ge 0}C_n\hat{I}_{2n}(z-\varepsilon_{\max})$$

for $d=1$ and

$$I_1(z)=-\pi\ln(z-\varepsilon_{\max})\sum_{n\ge 0}C_n(\varepsilon_{\max}-z)^n+\sum_{n\ge 0}C_n\tilde{I}_{2n+1}(z-\varepsilon_{\max})$$

for$d=2$.Combiningthisrepresentation of$I_1(z)$withthe expansion (17) of$I_2(z)$ andtakinginto account (18) and (19) we obtainthe thesis ofthe propositionwith$\delta:=\min\{\gamma/2,r_1/2\}\in(0,1/2)$.■

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